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SIMULATION OF RHEOMETRIC FLOW OF A NEWTONIAN FLUID BY THE METHOD OF FINITE ELEMENTS

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Flow between two plates is investigated numerically in the phase of reaching a steady-state value for a Newtonian fluid. Certain features of the application of the method of finite elements are discussed. The time of reaching the steady state by the flow is determined, which is constant for a wide range of parameters and fluids. It is noted that the results obtained can serve as the "point of departure" for comparing the considered flow with similar flows of viscoelastic and viscoplastic fluids and that the difference between the times of reaching the steady state by the flows can be used to evaluate the degree of non-Newtonian behavior of one fluid or another or of the model.

Flow between two parallel plates plays an important role in experimental rheology [1-2]. When a flow reaches the steady state, it allows one to record the so-called rheometric curves, i.e., the dependences of viscosity on the rate of shear.

However, for non-Newtonian fluid flows in the steady state it is impossible to determine such important properties as the presence of "memory" and the difference of normal stresses [2] that manifest themselves especially substantially in fast nonstationary flows. Moreover, any realizable steady-state flow must have a phase of reaching this state, and the behavior of the fluid can depend on the character of the phase and on the way in which the non-Newtonian properties manifest themselves in it. Therefore, investigation of this phase, in particular, in comparison with a similar phase of a Newtonian fluid, which is investigated in the present work, can be of great interest.

Formulation of the Problem. We introduce the Cartesian coordinate system, directing the x and y axes parallel and perpendicularly to the plates, respectively. Assuming that the y component of the velocity is equal to zero and that all the characteristics of the flow do not change along the x axis, the equations of the hydrodynamics of an incompressible fluid are reduced to the equation

$$\frac{\partial u}{\partial t}=\frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2},$$

where $\mathrm{Re} = \rho UH/\mu$, $t = t^*U/H$, $0 \le y = y^*/H \le 1$, and $u = u^*/U$.

Initial conditions: $u(y, 0) = 0$; *boundary conditions:* the process of smooth acceleration of the upper plate from zero velocity to the asymptotic velocity U will be described by the function

$$u_{\mathrm{pl}}=\frac{\alpha^* t^*}{1+\alpha^* t^*}. \quad (1)$$

Evidently, the parameter α^* characterizes the rate of build-up of the acceleration.

For the dimensionless velocity $u(1, t) = \alpha t/(1 + \alpha t)$, where $\alpha = \alpha^* H/U$. The lower plate is at rest: $u(0, t) = 0$.

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We will make the substitution $t \to \alpha t$ and write the problem in the form

$$\frac{\partial u(y,t)}{\partial t} = M \frac{\partial^2 u(y,t)}{\partial y^2}, \tag{2.1}$$

$$u(y,0) = 0, \tag{2.2}$$

$$u(0,t) = 0, \tag{2.3}$$

$$u(1,t) = \frac{t}{1+t}. \tag{2.4}$$

Thus, the entire class of the flows considered can be described by relations (2.1)-(2.4) with variation of only one parameter $M = 1/\alpha\mathrm{Re}$.

Method of Solution. This problem can be solved, in principle, analytically by means of the operational method [3]. However, the practical use and analysis of this kind of solution would be associated with the necessity of computer calculations of series and integrals. In the present work, the problem is solved directly by numerical simulation by means of the method of finite elements. In addition to the consideration of the main problem, this made it possible to use this simple example to investigate certain features of this method.

Let us apply the method of finite elements in two modifications [4]: with the use of linear finite elements $y_1 - y_2$ with the functions

$$W_1 = \frac{y - y_2}{y_1 - y_2}, \quad W_2 = \frac{y_1 - y}{y_1 - y_2}$$

and with the use of quadratic Lagrangian finite elements $y_1 - y_2 - y_3$ with the functions

$$W_1 = \frac{(y - y_2)(y - y_3)}{(y_1 - y_2)(y_1 - y_3)}; \; W_2 = \frac{(y_1 - y)(y - y_3)}{(y_1 - y_2)(y_2 - y_3)}; \; W_3 = \frac{(y_1 - y)(y_2 - y)}{(y_1 - y_3)(y_2 - y_3)}.$$

The application of the method of finite elements and the approximation of the time derivative in (2.1) according to Crank–Nicholson gives a system of equations, which precisely is solved numerically for the quantities u_k^{n+1}, $i = 1, ..., N-2$, by the method of successive upper relaxation

$$\left(A_{ik} + \frac{M\Delta t}{2} B_{ik}\right) u_k^{n+1} = \left(A_{ik} - \frac{M\Delta t}{2} B_{ik}\right) u_k^n, \tag{3}$$

Here $A_{ik} = \int_0^1 W_i W_k dy$ (Gram's matrix, which is symmetric and positive definite [5]) and $B_{ik} = \int_0^1 \frac{\partial W_i}{\partial y} \frac{\partial W_k}{\partial y} dy$.

The maximum rate of convergence of the iterations in virtually all the computations described below was attained at the relaxation parameter $\omega = 1.6$.

Testing. To investigate the convergence of the solution of the finite-element formulation of the problem, comparison was made of the numerical solution and particular solution of Eq. (2.1):

$$u(y,t) = \exp(-Mt) \sin y.$$

Naturally, instead of conditions (2.2)-(2.4) we prescribed

$$u(y,0) = \sin y, \tag{2.2$'$}$$

TABLE 1. Norms for Test Solution with Linear Finite Elements

Δt	N				
	21	61	101	141	201
0.0005	$3.84 \cdot 10^{-5}$	$4.42 \cdot 10{-}6$	$9.71 \cdot 10^{-6}$	$1.85 \cdot 10^{-5}$	$8.53 \cdot 10{-}5$
0.001	$3.98 \cdot 10^{-5}$	$4.65 \cdot 10{-}6$	$1.09 \cdot 10^{-5}$	$3.08 \cdot 10^{-5}$	$1.11 \cdot 10{-}4$
0.01	$7.92 \cdot 10^{-5}$	$3.82 \cdot 10^{-5}$	$2.92 \cdot 10{-}5$	$5.92 \cdot 10^{-5}$	0.00012
0.03	0.00052	0.00046	0.00044	0.00043	0.00033
0.05	0.0015	0.0014	0.0014	0.0014	0.0012

TABLE 2. Norms for Test Solution with Quadratic Finite Elements

Δt	N				
	11	31	51	71	101
0.0005	$1.80 \cdot 10^{-6}$	$4.29 \cdot 10{-}6$	$1.10 \cdot 10^{-5}$	$1.85 \cdot 10{-}5$	0.00013
0.001	$1.78 \cdot 10^{-6}$	$7.24 \cdot 10{-}6$	$1.55 \cdot 10^{-5}$	$5.82 \cdot 10^{-5}$	0.00015
0.01	$3.96 \cdot 10^{-5}$	$3.65 \cdot 10{-}5$	$3.67 \cdot 10^{-5}$	$7.47 \cdot 10{-}5$	0.00016
0.03	0.00047	0.00046	0.00042	0.00038	0.00028
0.05	0.0014	0.0014	0.0013	0.0013	0.0012

$$u(0, t) = 0, \tag{2.3$'$}$$

$$u(1, t) = \exp(-Mt) \sin 1 . \tag{2.4$'$}$$

Tables 1 and 2 present the magnitudes of the norm of the difference between the numerical and exact solutions when applying linear and quadratic finite elements in calculating up to $t = 1$ at $M = 10$ with uniform splitting (here, by the norm we mean $\max\limits_{k,n} |u_k^{n\text{pr}} - u_k^{n\text{num}}|$).

One can note that the use of quadratic finite elements is much more efficient: in the similar cells of the tables the values of the norms are commensurable, but in this case the number of nodes in the quadratic case is approximately half as small (more precisely, the quantities of nodes for the linear and quadratic finite elements are in the ratio $2N - 1/N$). Further, it is possible to state a good accuracy with the average frequency of division (11-51 points in Table 2). A finer grid (up to 101 points in Table 2) does not lead to higher accuracy.

We propose one of the possible reasons for this. As can easily be seen from the form of the finite-element functions and integrals, the norms of the matrices are $\|A\| \sim \Delta/$ and $\|B\| \sim 1/\Delta$. Further, it is not difficult to see that B is symmetric and "nonnegative" definite, i.e.,

$$\forall \bar{a} \neq 0 \sum_{i,k} B_{ik}\, a_i\, a_k \geq 0 .$$

Then the overall matrix of the left-hand side of (3) is evidently symmetric and positive definite (precisely this is in essence the "theoretical" basis for applying the SOR-method for solving system (3) [6]). Moreover, the matrix B is degenerate. Indeed, Eq. (2.1) has the trivial solution $u(y, t)$ = const; then (3) must admit the solution u_k^{n+1} = const; here

$$\sum_k B_{ik}\, u_k^{n+1} = \text{const} \sum_k B_{ik} = 0 ,$$

TABLE 3. Parameters Used in Investigation of Flow for Specific Fluids

Fluid	μ, Pa·sec	Re	α^*, sec^{-1}	M
Water	10^{-3}	100	5	$8 \cdot 10^{-3}$
Glycerin	1.5	0.067	5	12
High-pressure polyethylene	$\cong 10^5$	$\cong 10^{-6}$	5	$\cong 8 \cdot 10^5$

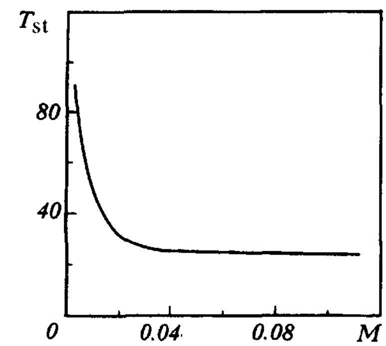

Fig. 1. Dependence of the dimensionless time of attaining the stationary state by the flow on the parameter M.

which is the condition for the linear dependence of the columns B. Therefore, the matrix B has at least one zero eigenvalue. In view of the estimations of the norms of the matrices, all the nonzero eigenvalues increase with N. It is not difficult to come to a conclusion about the decrease in the number of conditionality (ratio of the minimum and maximum eigenvalues) for the matrix of the left-hand side of (3), which probably influences the quality of solution [4].

This assumption was confirmed by the fact that at large N the number of iterations needed to obtain a solution increased substantially (up to 150-200) and, consequently, so did the time of calculation. Thus, on the basis of this test solution it was assumed justifiable to select N within the range 11-51 in the main version of the problem.

Parameters. Determination of the Range of the Values of M. For an actual rheometric experiment it is advisable to consider it reasonable to take the following values: H = 0.5 cm and U = 2 cm/sec; further, let us agree to consider the time of attaining the steady state by the upper plate the period during which its acceleration decreases to the fraction $\varepsilon = 1.5 \cdot 10^{-3}$ from the initial acceleration (ε was selected from consideration of the optimum time of calculation). From Eq. (2.4) we obtain that this time is $T_{st}^{pl} \cong 24.8$; for the dimensional time, if $\alpha^* = 5$ sec^{-1}, $T_{st}^{pl} \cong 5$ sec, which is acceptable from the practical point of view. The corresponding parameters for three fluids are presented in Table 3.

Evidently, to describe flows of a wide range of fluids (from virtually Newtonian (water) to viscoelastic (polymer melts)) it is necessary to assign the range $M = 10^{-3}$–10^{-5}.

Results and Discussion. The time of attainment of the stationary regime in comparison with the time T_{dev}^{pl} can be considered as one of the most important characteristics of a flow. It was determined from the condition that at this time the norm of the time derivative $\max\limits_k \left|\dfrac{\partial u_k^n}{\partial t}\right|$ did not exceed ε. The results of the determination of this time are presented in Fig. 1.

As was expected, at small values of M (smaller than 0.04) in a number of cases in the zone of a sharp change of velocity near the upper plate there were oscillations of the numerical solution on the first several time layers. However, it was not difficult to overcome them (by increasing somewhat the number of points and (especially) by bunching the splitting near the upper boundary and decreasing the time step (thereafter the step gradually increased with approach to the stationary regime). A conclusion can be drawn from the graph that only at small values of M does the time of attainment of the stationary state by the flow differ noticeably from the time of acceleration of the upper plate $T_{\mathrm{dev}}^{\mathrm{pl}}$. At larger M (up to 10^5) this time is close to the latter one.

Similar dependences calculated or measured experimentally in the case of non-Newtonian fluids in comparison with the given dependence could probably give information about the extent of the manifestation of non-Newtonian properties by the fluid for different reasons (viscoplasticity, viscoelasticity).

NOTATION

t^*, y^*, u^*, t, y, and u, dimensional and dimensionless values of the time, vertical coordinate, and x-component of the fluid velocity; ρ and μ, density and dynamic viscosity of the fluid; Re, Reynolds number; u_{pl}, velocity of the upper plate; U, velocity of the upper plate in constant motion; H, width of the gap between the plates; α^* and α, dimensional and dimensionless values of the parameter that characterizes the rate of increase of the acceleration of the upper plate; M, dimensionless parameter; FEs, finite elements; i and k, numbers of finite-element nodes; W_1, W_2, W_3, and W_i, finite-element functions; y_1, y_2, and y_3, nodes contained in the instantaneous finite element; u_i^n, nodal value of the velocity on the n-th time layer; N, number of nodes of finite-element splitting; A_{ik} and B_{ik}, elements of the matrices A and B composed of internal finite-element products; $\bar{a}$, vector; Δt, numerical time step; Δ, smallness of the finite-element splitting; $T_{\mathrm{st}}^{\mathrm{pl}}$ and $T_{\mathrm{st}}^{\mathrm{pl}}$, dimensional and dimensionless values of the time of development of the motion of the upper plate; ε, small parameter that determines the time of attainment of the stationary regime by the motion of the upper plate and fluid.

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